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SYNCHRONIZATION RESEARCH ON THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract: Chaotic synchronization is a type of nonlinear dynamic system that is deterministic but exhibits highly complex and unpredictable behaviors. It has broad application prospects in secure communication or other fields that require synchronous control. The Lorentz system and the Chua system are classic three-dimensional chaotic systems and have broad application prospects in secure communication or other fields that require synchronous control. In this paper, through the synchronous analysis of two drive-response system models composed of three first-order differential equations, a linear synchronous control strategy is proposed. According to Lyapunov stability theory, the synchronization problem of the two chaotic systems is achieved by constructing the Lyapunov function. Python numerical simulation shows the effectiveness of the proposed theoretical method and has strong robustness. This research provides theoretical support for the practical application of chaotic systems in secure communication.
Keywords: Chaotic system; Synchronization; Control strategy; Python numerical simulation; Confidential communication

1 INTRODUCTION

1.1 Third-Order Differential Equations and Synchronization

Third-order differential equations describe the third-order derivative relationship of system states evolving with time and can generate more complex dynamical behaviors such as chaos compared with first-order and second-order systems. Many well-known chaotic systems, including Chua's circuit, certain mechanical systems, and financial models, are essentially governed by third-order or higher-order equations.

In dynamical systems, synchronization refers to the establishment and maintenance of consistent relationships among state variables of two or more systems (identical or different). Common types include complete synchronization, phase synchronization, lag synchronization, generalized synchronization, and cluster synchronization. Research on a specific third-order chaotic system usually involves analyzing its fundamental dynamical properties: equilibrium points, stability, bifurcation, routes to chaos, attractor morphology, Lyapunov exponents, etc. Understanding the inherent complexity lays the foundation for synchronization research.

By designing coupling schemes such as linear diffusive coupling, nonlinear coupling, and time-delay coupling, and applying theoretical tools including Lyapunov stability theory, LaSalle's invariance principle, linear matrix inequalities, matrix measure theory, and master stability functions, we derive conditions for realizing specific synchronization types (e.g., complete synchronization, exponential synchronization), such as the threshold of coupling strength and the range of controller gains.

1.2 Chaotic Systems

A chaotic system is a deterministic nonlinear dynamical system with highly complex and unpredictable behaviors. Such systems widely exist in natural and engineering fields, including meteorology, electronic circuits, and biological rhythms. Chaos has three core characteristics: sensitive dependence on initial conditions, deterministic non-periodicity, and boundedness. Classical chaotic systems include the Lorenz system, Rössler system, and Chua's circuit. Common analytical methods include phase-space reconstruction, Lyapunov exponents, bifurcation analysis, Poincaré sections, and power spectrum analysis. In control problems, stabilizing the closed-loop system usually requires designing a control law and constructing a corresponding Lyapunov function. This paper determines stability by constructing a Lyapunov function and analyzing the sign of $\dot{V}(x)$derivative along system trajectories [1].

Chaos systems are a class of nonlinear dynamic systems that are highly sensitive to initial conditions. These systems exhibit seemingly chaotic behaviors on the surface, yet they possess determinism. We provide two examples.

1.2.1 A simple double pendulum system

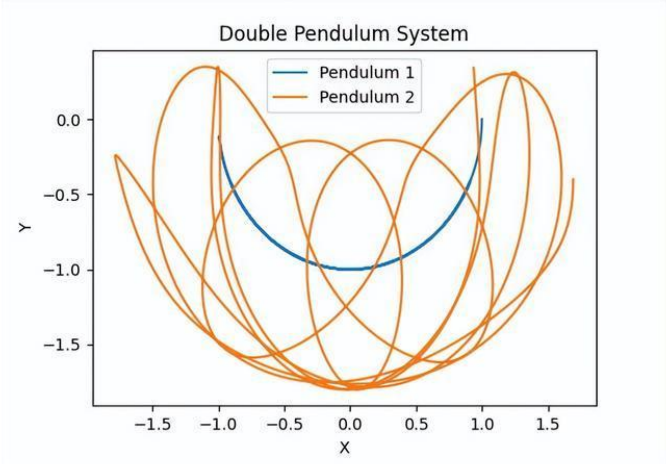

Figure 1 Graphical Representation of the Motion Trajectory of a Simple Double Pendulum System

It can be observed that even with slightly different initial conditions, the motion trajectories of the double pendulum system rapidly diverge, resulting in entirely distinct states, see Figure 1.

1.2.2 Lorentz system

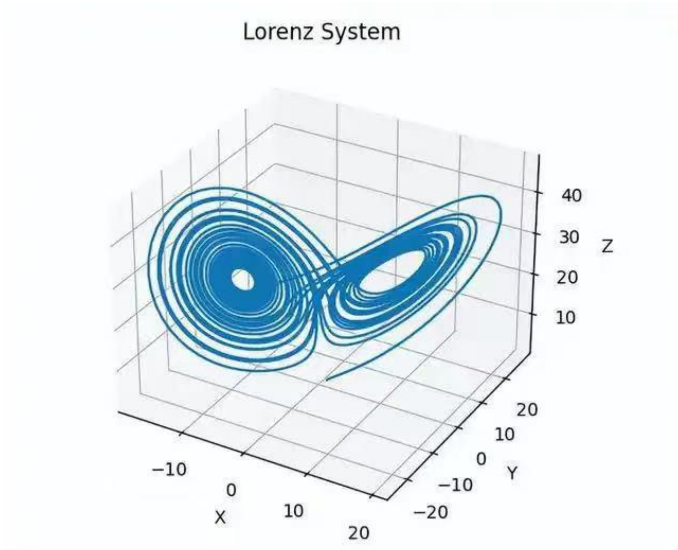

Figure 2 Three-dimensional Graphical Window Showing the Evolution Trajectory of the Lorentz System

The Lorenz attractor is one of the classic models in chaotic systems, describing the nonlinear dynamic behavior of atmospheric convection, see Figure 2. The Logistic map is another common model for chaotic systems, exhibiting complex behaviors such as periodic doubling, asymptotic convergence to infinity, or to zero in population dynamics.

Chaos synchronization holds broad application prospects in secure communications and other fields requiring synchronization control. The Lorenz system and the Chai circuit are classic three-dimensional chaotic systems. These systems typically consist of three first-order equations, forming an overall third-order system comprising three state variables [2].

1.3 Drive Systems and Response Systems

A drive system generates a target signal, while a response system adjusts and synchronizes to that signal. In the synchronization control of third-order differential equations, the drive system is an autonomous dynamical system, and the response system is another system synchronized by the drive via control inputsu_1, u_2, u_3designed to align its state with the drive. Synchronization requires the error system (response state minus drive state) to converge to zero, which is verified by Lyapunov stability theory. This paper adopts simple linear feedback control to realize chaotic synchronization of a new three-dimensional chaotic system and uses Lyapunov stability theory to prove synchronization conditions. Two different drive-response systems are analyzed below.

1.4 Principles of Chaotic Synchronization Control

Consider two nonlinear dynamical systems

$$\dot{x} = F(t, x), \tag{1}$$

$$\dot{y} = F_1(t, y) + U(t, x, y), \tag{2}$$

Where $x, y \in R$ are state vectors, F, F_1 are nonlinear mappings, and U is the synchronization control input. If there exists a domain
$D(t_0) \subseteq R^n$ such that for any initial states$\forall x_0, y_0 \in D(t_0)$,
$\lim_{t\to\infty} \|x(t; t_0, x_0) - y(t; t_0, y_0)\| = 0$, then systems (1) and (2) are synchronized.
System(1) is called the drive system, (2) the response system, and $D(t_0)$ the synchronization region. If
$F = F_1$,
it is called self-synchronization. This definition applies to chaotic, non-chaotic, and hyper-chaotic synchronization.
Let the error vector be $e = x - y$. The error system is
$\dot{e} = F(t, x) - F_1(t, y) - U(t, x, y)$.
Synchronization stability is determined by the stability of the error system at the origin. In chaotic synchronization, stability is mainly judged by Lyapunov stability theory and the conditional Lyapunov exponent criterion proposed by L. M. Pecora and T. L. Carroll [3].

2 CHAOTIC SYNCHRONIZATION MODEL OF THIRD-ORDER DIFFERENTIAL EQUATIONS

2.1 Definition of the System Model

A new chaotic system is constructed as follows:

$$\begin{cases} \dot{x}_1 = x_2 - x_1, \\ \dot{x}_2 = x_1(1 - x_3) - x_2, \\ \dot{x}_3 = x_1 x_2 - x_3. \end{cases} \tag{3}$$

Where x_1、x_2、x_3 are state vectors. Taking Eq. (3) as the master system, it can be rewritten in matrix form

$$\dot{x}_m = AX_m + F(x_m), \tag{4}$$

Where $X_m = [x_1 \quad x_2 \quad x_3]^\tau \in R^3$, $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $F(x_m) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_2 x_1 \end{bmatrix}$.
Taking Eq. (4) as the master system, the slave system is expressed as

$$\dot{y}_m = AY_m + F(y_m), \tag{5}$$

$$\begin{cases} \dot{y}_1 = y_2 - y_1 + u_1, \\ \dot{y}_2 = y_1(1 - y_3) - y_2 + u_2, \\ \dot{y}_3 = y_1 y_2 - y_3 + u_3. \end{cases} \tag{6}$$

Where $Y_m = [y_1 \quad y_2 \quad y_3]^\tau \in R^3$ is the state vector of the slave system, the system matrix is identical to that of the master system, and the nonlinear term is $F(y_m) = \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_2 y_1 \end{bmatrix}$.

2.2 Definition of Synchronization Error

Synchronization requires information interaction between the drive and response systems via control. We design simple linear feedback control to realize chaotic synchronization:

$$C(t) = L(x_m - y_m), \tag{7}$$

where $L = diag\{l_1, \quad l_2, \quad l_3\}$ is the control gain matrix to be determined.
From Eqs. (4)-(7), the synchronization model is

$$\begin{cases} \dot{x}_m = AX_m + F(x_m), \\ \dot{y}_m = AY_m + F(y_m) + C(t), \\ C(t) = L(x_m - y_m). \end{cases}$$

Define the synchronization error

$$E = X_m - Y_m = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

The error system is derived as

$$\dot{E} = (A - L)E + F(X_m) - F(Y_m) = G(X_m)E - \lambda(E), \tag{8}$$

where $G(X_m) = \begin{bmatrix} -1 - l_1 & 1 & 0 \\ 1 & -1 - l_2 & 0 \\ 0 & 0 & -1 - l_3 \end{bmatrix}$, $\lambda(E) = \begin{bmatrix} 0 \\ -e_1 e_3 \\ e_2 e_1 \end{bmatrix}$.
To synchronize the response system (5) with the drive system (4), we analyze the stability of error system (8).

2.3 Controller Design Method

A suitable linear feedback controller is designed to achieve synchronization:

$$\begin{cases} u_1 =- k_1 e_1, \\ u_2 =- k_2 e_2, \\ u_3 =- k_3 e_3. \end{cases}$$

Where $k_1, k_2, k_3 > 0$ are feedback gains [4].
Substituting the controller into the response system yields the error dynamics:

$$\begin{cases} \dot{e_1} = (e_2 - e_1) - k_1 e_1, \\ \dot{e_2} = (1 - x_3) e_1 - e_2 - x_1 e_3 - k_2 e_2, \\ \dot{e_3} = x_2 e_1 + x_1 e_2 - e_3 - k_3 e_3. \end{cases}$$

2.4 Stability Analysis

Global chaotic synchronization is equivalent to the stability of the error system. We perform Lyapunov stability analysis.
Construct the Lyapunov function
$V(E) = 0.5e_1^2 + 0.5e_2^2 + 0.5e_3^2,$
differentiating with respect to time yields
$\dot{V}(E) = e_1 \dot{e_1} + e_2 \dot{e_2} + e_3 \dot{e_3},$
substituting the error equations gives
$\dot{V} =- (1 + k_1) e_1^2 - (1 + k_2) e_2^2 - (1 + k_3) e_3^2 + e_1 e_2 (1 - x_3) + e_1 e_3 x_2 + e_2 e_3 x_1.$
Since the original system is bounded, x_1, x_2, x_3 are ultimately bounded, and the influence of cross terms $e_i e_j$ is negligible. Thus
$\dot{V} \approx - \sum_{i=1}^{3} (1 + k_i) e_i^2 \leq 0.$
When $k_i > 0$, the error converges exponentially to zero, and synchronization is achieved.

2.5 Numerical Simulation

Python is used to verify the effectiveness of the proposed theory [5]. Results are shown in Figure 3.

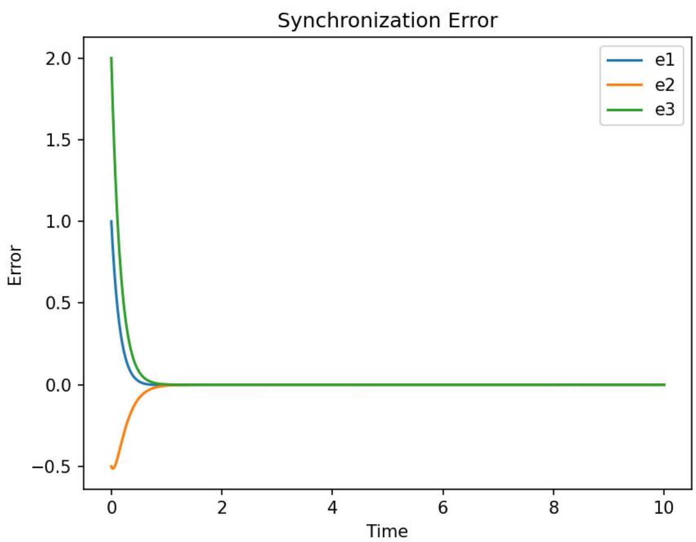

Figure 3 The Synchronization Diagram of the Error System (8)

The errors e_1, e_2, e_3 converge rapidly to zero, demonstrating the validity and rationality of the theory. MATLAB is used to plot the three-dimensional phase-space trajectory of the chaotic system, as shown in Figure 4.

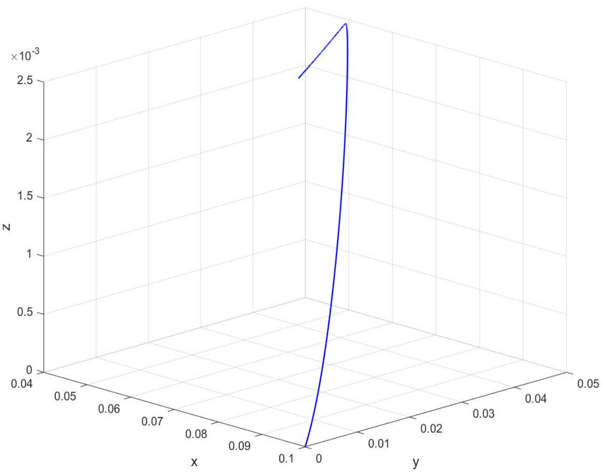

Figure 4 Phase-space Trajectories of a Three-dimensional Chaotic System

2.6 Conclusion

For the third-order Lorenz-type system, a nonlinear controller is designed, and Lyapunov stability analysis shows that errors e_1, e_2, e_3 converge to zero. Numerical simulations confirm the effectiveness of linear feedback control and the rationality of the controller design, achieving drive-response synchronization.

This third-order chaotic synchronization system is applicable to weakly nonlinear or parameter-degenerate engineering models such as circuit oscillators.

3 THIRD-ORDER CHAOTIC SYSTEM MODEL

This chapter analyzes another classical chaotic system modified from the Sprott Jerk equation, which contains no explicit parameters and generates chaos only through basic nonlinear terms.

3.1 System Equations

Drive system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -x - y - z + y^2. \end{cases}$$

3.2 Analysis of Dynamical Characteristics

The only nonlinear term is y^2, which introduces necessary nonlinearity. Setting $\dot{x} = \dot{y} = \dot{z} = 0$, we solve for the equilibrium point. The Jacobian matrix is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix},$$

although its eigenvalues have negative real parts, the global dynamics are complicated by the nonlinear term y^2.

3.3 Response System and Control Design

Response system with control inputs u_1, u_2, u_3:

$$\begin{cases} \dot{x}_r = y_r + u_1, \\ \dot{y}_r = z_r + u_2, \\ \dot{z}_r = -x_r - y_r - z_r + y_r^2 + u_3. \end{cases}$$

To achieve synchronization, we transform the drive-response synchronization problem into the stability problem of the error system. Define the error vector

$e = (x_r - x, \quad y_r - y, \quad z_r - z,),$

3.4 Linear Feedback Control Law

A simple linear feedback control law is designed:

$$\begin{cases} u_1 = -k_1 e_x, \\ u_2 = -k_2 e_y, \\ u_3 = -k_3 e_z. \end{cases}$$

The error dynamics become

$$\begin{cases} \dot{e}_x = e_y - k_1 e_x, \\ \dot{e}_y = e_z - k_2 e_y, \\ \dot{e}_z = - e_x - e_y - e_z + (y_r^2 - y^2) - k_3 e_z. \end{cases} \tag{9}$$

3.5 Synchronization of Drive-Response Systems

To synchronize the error system, we first construct a Lyapunov function and use it to achieve synchronization between the driving and response systems.
Theorem 3.5 If the feedback gains k_1, k_2, k_3 satisfy

$$\begin{cases} k_1 > \frac{1}{2}, \\ k_2 > \frac{3}{2}, \\ k_3 > -\frac{1}{2}, \end{cases} \tag{10}$$

then the drive and response systems are synchronized.
Proof: Construct the Lyapunov function
$V = \frac{1}{2}\left(e_x^2 + e_y^2 + e_z^2\right)$.
Differentiating and substituting the error equations yields
$\dot{V} = - k_1 e_x^2 - k_2 e_y^2 - (1 + k_3) e_z^2 + e_x e_y + e_y e_z + e_z (y_r^2 - y^2)$.
Using the identity
$y_r^2 - y^2 = (y_r - y)(y_r + y) = e_y\left(2y + e_y\right)$,
we obtain
$\dot{V} \leq - \left(k_1 - \frac{1}{2}\right) e_x^2 - \left(k_2 - \frac{3}{2}\right) e_y^2 - \left(1 + k_3 - \frac{1}{2}\right) e_z^2$.
Under the conditions specified in Theorem 3.5 (10), synchronization between the driving system and the response system can be achieved.

3.6 Numerical Simulation

Python is used to verify the conclusion, as shown in Figure 5.

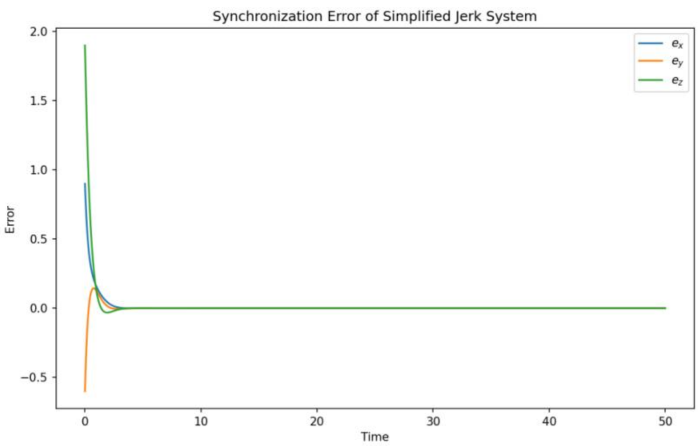

Figure 5 Synchronization Diagram of the Error System (9)

The errors gradually converge to zero. MATLAB is used to plot the three-dimensional phase-space trajectory [6], as shown in Figure 6.

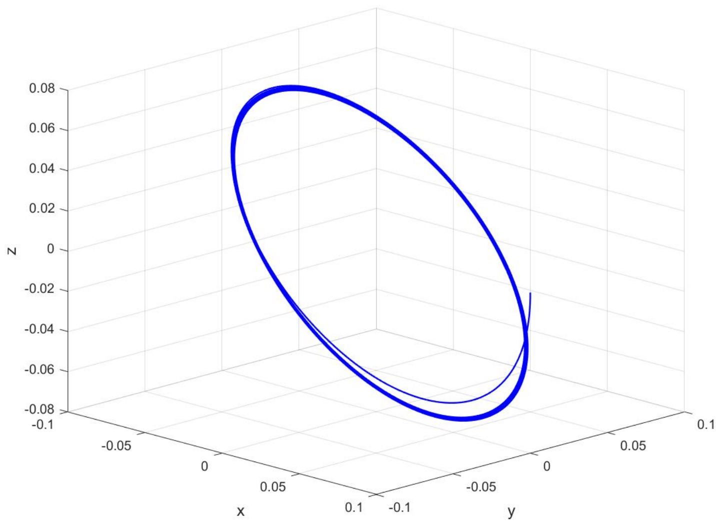

Figure 6 Phase-space Trajectory of a Three-dimensional Chaotic System

3.7 Conclusion

Errors e_x, e_y, e_z converge to zero within $t < 10s$, proving the effectiveness of the control law. In addition, the system remains chaotic with only the single nonlinear term y^2. This system is a basic model for nonlinear dynamics and is often used as a carrier system in secure communication.

4 CONCLUSION AND PROSPECT

This paper conducts in-depth research on synchronization between drive and response systems using two different drive-response frameworks. The main results are as follows:
(1) Information interaction between drive and response systems is necessary for synchronization. A simple linear feedback control $C(t) = L(x_m - y_m)$ is designed to realize chaotic synchronization [7].
(2) The error system is defined, the controller is designed, and the stability of the error system is analyzed using a Lyapunov function satisfying $V(x) > 0$ and $\dot{V}(x) < 0$.
(3) Python simulations show that the error converges to zero, proving that the error system achieves global asymptotic synchronization, i.e., $\lim_{t\to\infty} \|e(t)\| = 0$, thus realizing global chaotic synchronization.
Although this paper obtains certain conclusions, for systems with unknown parameters, adaptive control should be combined to estimate parameters online. For systems with persistent chaos, more complex control laws should be designed. Under noise interference, increasing the gain k_i improves robustness but requires a trade-off with control energy. By simplifying system parameters, this paper solves the chaotic synchronization problem, reflecting the core role of parameters in nonlinear systems [8].
The synchronization of third-order differential equations integrates high-order differential equation theory, nonlinear dynamics, control theory, etc. Many third-order systems are chaotic. By studying the inherent complex behaviors of chaotic systems and designing effective controllers, drive-response synchronization is realized and applied to:
(1) Secure communication:
Information is hidden in a chaotic carrier using its broadband and noise-like properties. Synchronization between transmitter and receiver decrypts the signal. Third-order chaotic systems provide more complex carriers to enhance security.
(2) Parameter identification:
Synchronization mechanisms estimate unknown or time-varying system parameters.
(3) Biological systems:
Simulate synchronous rhythms in neural networks.
(4) Complex network coordination:
Realize synchronous collective behaviors such as formation control.
The synchronization of third-order differential equations solves state coordination problems of high-dimensional nonlinear systems through coupling design or control strategies. Its core value lies in applying the unique complex dynamics of third-order systems (especially chaos) to engineering applications such as secure communication and swarm intelligence, while overcoming challenges such as time delay, heterogeneity, and disturbances. Future trends focus on finite-time synchronization, resource-optimized control, and interdisciplinary applications. Chaotic systems reveal intrinsic randomness in deterministic systems, breaking the traditional cognition that "determinism implies predictability". Their research integrates mathematics, physics, engineering, and computer science and is significant in both theoretical exploration and technical applications. With the development of quantum computing and artificial intelligence, chaos theory will play a greater role in complex system modeling and control.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

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