

CONSTRUCTION AND GEOMETRIC PROPERTIES OF N-QUASI-BEZIER-CURVES WITH MULTIPLE PARAMETERS

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Abstract: By using novel recurrence relation, we define a Quasi-Bernstein-basis of degree n with multiple parameters. Next, based on Quasi-Bernstein-basis, constructs n-Quasi-Bézier-curves (n-QB-curves) and systematically investigates their core geometric properties and computational methods. We propose an angle-cutting algorithm specifically designed for QB-curves, deriving the principles governing local shape adjustments and tension control through parameter manipulation. The length quantization method is established to analyze the curve approximation effect, and the proposed n-Quasi-Bézier-curves have better approximation performance than the traditional method. It is proved that the global or local shape modification of the joining curve can be achieved by adjusting the parameters when the polygon is fixed. The research shows that n-Quasi-Bézier-curves can approximate the control polygon without any degree increase, which has the characteristics of high efficiency and good shape control ability, and provides a new method and idea for the curve design in the field of geometric modeling.

Keywords: Multiple shape parameters; Corner cutting algorithm

1 INTRODUCTION

In computer-aided geometric design (CAGD) and related fields, Bézier curves have become one of the most fundamental tools for curve and surface modeling due to their simple algebraic form, geometric intuitiveness, and numerical stability [1-7]. However, classical Bézier curves suffer from a significant limitation: the shape of a Bézier curve is globally determined by its control polygon, and local shape adjustment is impossible without changing the control points or increasing the degree [2-4]. To overcome this drawback, researchers have introduced various extensions, such as trigonometric Bézier curves [8-11], quasi-Bernstein bases with shape parameters [1-7], and multi-parameter Bézier-like curves [2-6]. These extensions often provide additional degrees of freedom for shape tuning, but many of them either require degree elevation to achieve better approximation or lack efficient computational algorithms [3-5].

Recently, the concept of quasi-Bernstein bases with multiple parameters has attracted increasing attention. By introducing extra parameters without raising the degree, one can achieve flexible local shape control and tension effects [6-7]. Nevertheless, existing methods still face challenges in balancing computational efficiency, approximation ability, and smooth connection conditions [6-13]. Therefore, constructing a new type of curve that can infinitely approach the control polygon without degree elevation, while supporting simple and stable corner cutting algorithms and flexible geometric continuity conditions, remains an important research problem.

In this paper, we propose a novel family of n-Quasi-Bézier-curves (QB-curves) based on a recursively defined quasi-Bernstein basis with multiple parameters. The main contributions are as follows:

We present an efficient angle-cutting algorithm (a de Casteljau-type method) for evaluating QB-curves of arbitrary degree, which is both numerically stable and computationally simple [11-14].

We systematically investigate the local shape adjustment and tension control effects of the parameters. It is shown that, for a fixed control polygon, increasing the parameters makes the QB-curves move towards the control polygon, achieving infinite approximation without degree elevation [2-14].

A length-based quantitative method (L-Method) is developed to measure the approximation performance. Comparative analysis with existing approaches.

The results show that the proposed QB-curves combine high computational efficiency, excellent shape control, and superior approximation performance, providing a powerful new tool for curve design in geometric modeling. The remainder of this paper is organized as follows. Section 2 defines the quasi-Bernstein basis and the QB-curves. Section 3 presents the corner cutting algorithm. Section 4 discusses the shape adjustment effect of parameters. Section 5 analyses the approximation performance using the L-Method.

2 DEFINITION OF QUASI-BÉZIER-CURVES

Definition 1 Given control points $P_{i,j} \in R^3 (i, j \in N, i + j = n, n \geq 2)$, and a domain $d = \{(u, v) | u + v = 1, u \geq 0, v \geq 0\}$. We call

$$R^n(u, v) = \sum_{i+j=n} P_{i,j} A_{i,j}^n(u, v), (u, v) \in d, \tag{1}$$

the Quasi-Bézier-curves (QB-curves for short) of degree n with $2\lceil n/2 \rceil$ parameters, where $A_{i,j}^n(u, v)$ are the QB-basis given in (2) $A_{i,j}^n(u, v) = f_{i-1,j}^{n-1}(u, v)B_{i-1}^{n-1}(u, v) + g_{i,j-1}^{n-1}(u, v)B_{i,j-1}^{n-1}(u, v)$.

Where

$$B_{x,y}^{n-1}(u, v) = \frac{(n-1)!}{x!y!} u^x v^y, (x, y \in N, x + y = n - 1) \tag{2}$$

3 CORNER CUTTING ALGORITHM

In this subsection, we give a corner cutting algorithm to efficiently and stably compute QB-curves. For the convenience of discussion, we will rewrite the QB-curves given in (1) as the following matrix form:

$$R(u, v) = (u \ v)_{1 \times 2} \times \begin{pmatrix} u & v & 0 \\ 0 & u & v \end{pmatrix}_{2 \times 3} \times \begin{pmatrix} u & v & 0 & 0 \\ 0 & u & v & 0 \\ 0 & 0 & u & v \end{pmatrix}_{3 \times 4} \times \dots \times \begin{pmatrix} u & v & 0 & \dots & 0 & 0 & 0 \\ 0 & u & v & \dots & 0 & 0 & 0 \\ 0 & 0 & u & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u & v & 0 \\ 0 & 0 & 0 & \dots & 0 & u & v \end{pmatrix}_{(n-1) \times n} \times H \times \begin{pmatrix} P_{0,n} \\ P_{1,n-1} \\ P_{2,n-2} \\ \vdots \\ P_{n-1,1} \\ P_{n,0} \end{pmatrix} \tag{3}$$

where H is a $n \times (n + 1)$ matrix. When n is odd, we have

$$H = \begin{pmatrix} \Omega_{n-1,0}^{n-1} & 1 - \Omega_{n-1,0}^{n-1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \Omega_{n-2,1}^{n-1} & 1 - \Omega_{n-2,1}^{n-1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Omega_{\frac{n+1}{2}, \frac{n-3}{2}}^{n-1} & 1 - \Omega_{\frac{n+1}{2}, \frac{n-3}{2}}^{n-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & u & v & \vdots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 - \Omega_{\frac{n-3}{2}, \frac{n+1}{2}}^{n-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \Omega_{1,n-2}^{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 - \Omega_{0,n-1}^{n-1} & \Omega_{0,n-1}^{n-1} \end{pmatrix} \tag{4}$$

and when n is even, we have

$$H = \begin{pmatrix} \Omega_{n-1,0}^{n-1} & 1 - \Omega_{n-1,0}^{n-1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \Omega_{n-2,1}^{n-1} & 1 - \Omega_{n-2,1}^{n-1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Omega_{\frac{n+1}{2}, \frac{n-1}{2}}^{n-1} & 1 - \Omega_{\frac{n+1}{2}, \frac{n-1}{2}}^{n-1} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 - \Omega_{\frac{n-1}{2}, \frac{n+1}{2}}^{n-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \Omega_{1,n-2}^{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 - \Omega_{0,n-1}^{n-1} & \Omega_{0,n-1}^{n-1} \end{pmatrix} \tag{5}$$

Where

$$\Omega_{x,y}^{n-1} = \begin{cases} \frac{u}{1 + \alpha_{x,y}^{n-1}(1-u)}, i - \lceil (n+1)/2 \rceil > 0, \\ \frac{v}{1 + \alpha_{x,y}^{n-1}(1-v)}, j - \lceil (n+1)/2 \rceil > 0, \end{cases} (x, y \in N, x + y = n - 1). \tag{6}$$

From the above discussion, in Figure 1, 2 and 3, we will offer the special form of the corner cutting algorithm of QB-curves of degree 3 and 4 to compute points $R^3(0.5, 0.5)$ and $R^4(0.5, 0.5)$, respectively. For convenience, we set the parameters involved in QB-curves as vector form, that is $(\alpha_{n-1,0}^{n-1}, \alpha_{n-2,1}^{n-1}, \alpha_{n-3,2}^{n-1}, \dots, \alpha_{0,n-1}^{n-1})$.

Figure 1 (a) describes the generating process for any point of the cubic QB-curves in detail, and Figure 1 (b,c and d) show specific examples of the corner cutting algorithm with different parameters. The parametric values of QB-curves given in Figure 1 (b,c and d) are $(\alpha_{2,0}^2, \alpha_{0,2}^2) = (0,0), (1,0)$ and $(0,1)$, respectively.

Figure 2 (a) describes the generating process for any point of the quartic QB-curves, and Figure 2 (b,c and d) show specific examples of the corner cutting algorithm with different parameters. The parametric values of QB-curves given in Figure 2 (b,c and d) are $(\alpha_{3,0}^3, \alpha_{2,1}^3, \alpha_{1,2}^3, \alpha_{0,3}^3) = (0,0,0,0), (0,10,10,4)$ and $(5,0,0,0)$ respectively.

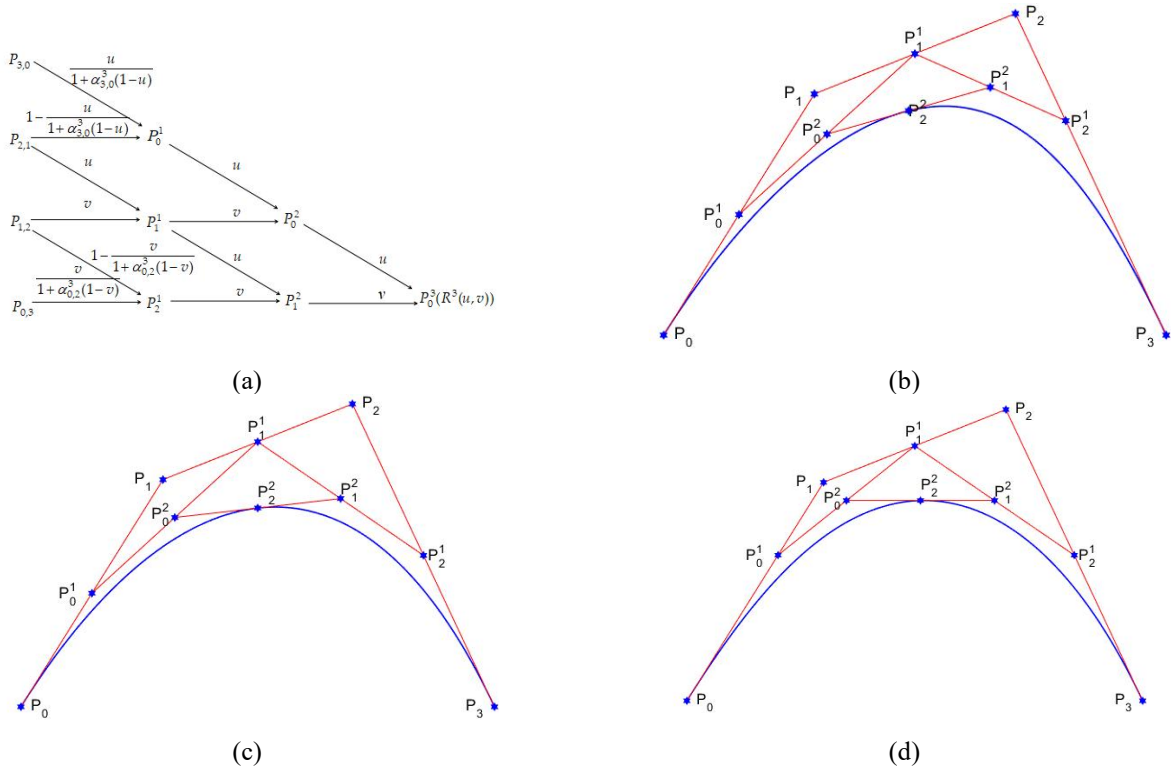


Figure 1 Corner Cutting Algorithm of QB-Curves of Degree 3

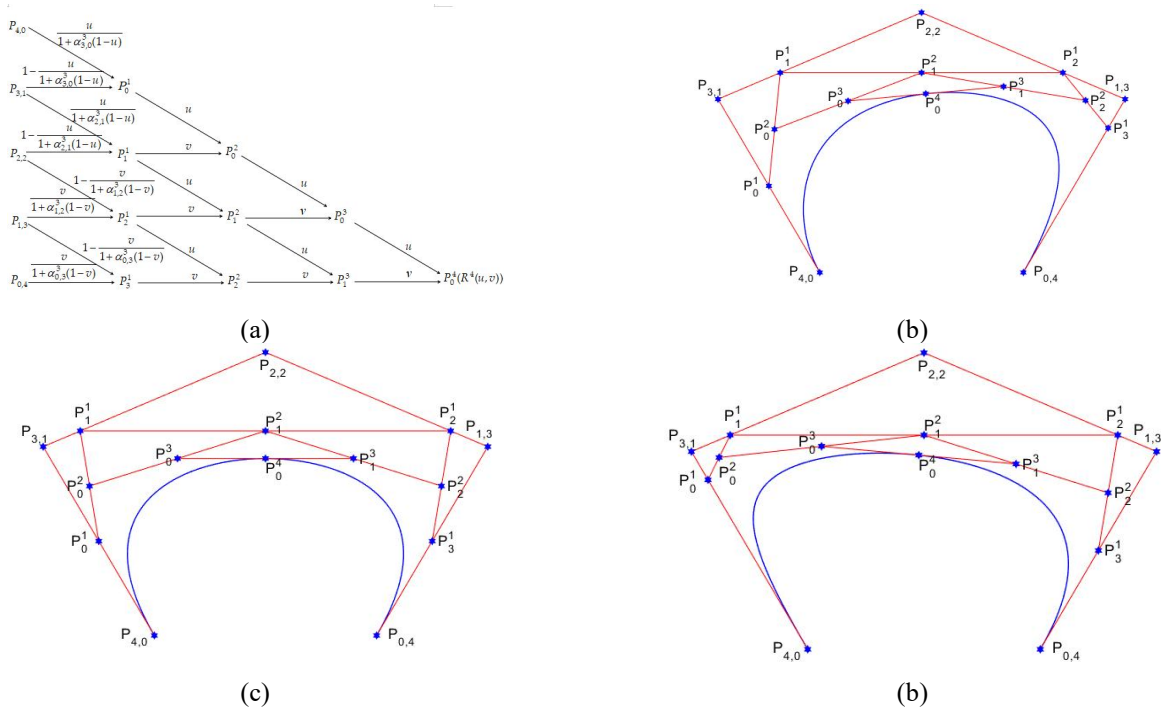


Figure 2 Turning Point Cutting Algorithm for Quadratic QB Curve

4 ADJUSTING EFFECT OF PARAMETERS ON QB-CURVES

Since the QB-basis given in (2) possesses $2\lfloor n/2 \rfloor$ different parameters $\alpha_{x,y}^{n-1}$, the shape of the related QB-curves could be adjusted by altering parameters. We will describe adjusting effect of the parameters on QB-curves in detail. The parameter $\alpha_{x,y}^{n-1}$ can locally modify shape of the QB-curves when given a fixed control polygon. In order to better describe this phenomenon, the QB-curve given in (1) can be rewritten as follows:

$$\begin{aligned}
 R^n(u, v) &= \sum_{i=1}^{(n-1)/2} C_{n-1}^{i-1} u^{i-1} v^{n-i} P_{i-1, n-i+1} + \sum_{j=1}^{(n-1)/2} C_{n-1}^{j-1} u^{j-1} v^{n-j} P_{j-1, n-j+1} + \\
 &C_{n-1}^{(n-1)/2} u^{(n-1)/2} v^{(n+1)/2} P_{(n-1)/2, (n+1)/2} + C_{n-1}^{(n-1)/2} u^{(n+1)/2} v^{(n-1)/2} P_{(n+1)/2, (n-1)/2} + \\
 &\sum_{i=0}^{(n-1)/2} C_{n-1}^i \frac{1}{1 + \alpha_{i, n-i-1}^{n-1} (1-v)} u^i v^{n-i} (P_{i-1, n-i} - P_{i+1, n-i-1}) + \\
 &\sum_{j=0}^{(n-1)/2} C_{n-1}^j \frac{1}{1 + \alpha_{n-j-1, j}^{n-1} (1-u)} u^{n-j} v^j (P_{n-j, j} - P_{n-j-1, j+1}),
 \end{aligned} \tag{7}$$

if n is odd, and

$$\begin{aligned}
 R^n(u, v) &= \sum_{i=1}^{n/2} C_{n-1}^{i-1} u^{i-1} v^{n-i} P_{i-1, n-i} + \sum_{j=1}^{n/2} C_{n-1}^{j-1} u^{n-j} v^{j-1} P_{n-j, j} + \sum_{i=0}^{(n/2)-1} C_{n-1}^i \frac{1}{1 + \alpha_{i, n-i-1}^{n-1} (1-v)} u^i v^{n-i} (P_{i, n-i} - P_{i+1, n-i-1}) \\
 &+ \sum_{j=0}^{(n/2)-1} C_{n-1}^j \frac{1}{1 + \alpha_{n-j-1, j}^{n-1} (1-u)} u^{n-j} v^j (P_{n-j, j} - P_{n-j-1, j+1})
 \end{aligned} \tag{8}$$

if n is even.

We first consider the case of odd. Thus, for any fixed i , from the formula (8), it follows that the parameter $\alpha_{i, n-i-1}^{n-1}$ could affect the QB-curves on edge $\overline{P_{i+1, n-i-1} P_{i, n-i}}$ when $i = 0, 1, \dots, n/2 - 1$ and parameters $\alpha_{n-j-1, j}^{n-1}$ could affect the QB-curves on edge $\overline{P_{n-j-1, j+1} P_{n-j, j}}$ when $j = 0, 1, \dots, n/2 - 1$, respectively. Furthermore, for any fixed i and $u, v \in [0, 1)$, the expression $1/[1 + \alpha_{i, n-i-1}^{n-1} (1-v)]$ and $1/[1 + \alpha_{n-j-1, j}^{n-1} (1-u)]$ will decrease with the increase of parameters $\alpha_{i, n-i-1}^{n-1}$ and $\alpha_{n-j-1, j}^{n-1}$, respectively, which means that the QB-curves move in the same direction of the edge $\overline{P_{i+1, n-i-1} P_{i, n-i}}$ and $\overline{P_{n-j-1, j+1} P_{n-j, j}}$ with the increase of the parameters $\alpha_{i, n-i-1}^{n-1}$ and $\alpha_{n-j-1, j}^{n-1}$ when $i = 0, 1, \dots, n/2 - 1$ and $j = 0, 1, \dots, n/2 - 1$, respectively. For the case of even, the moving behavior of the QB-curves generally same as the odd case.

In addition, the parameters $\alpha_{x,y}^{n-1}$ also serve as tension parameter. For arbitrarily single parameter $\alpha_{x,y}^{n-1}$ the expressions $1/[1 + \alpha_{n-j-1, j}^{n-1} (1-u)]$ and $1/[1 + \alpha_{i, n-i-1}^{n-1} (1-v)]$ meet the following rules

$$\frac{1}{[1 + \alpha_{n-j-1, j}^{n-1} (1-u)]} \rightarrow 0 (\alpha_{n-j-1, j}^{n-1} \rightarrow +\infty), \quad \frac{1}{[1 + \alpha_{i, n-i-1}^{n-1} (1-v)]} \rightarrow 0 (\alpha_{i, n-i-1}^{n-1} \rightarrow +\infty) \tag{9}$$

Thus, from formulas (8) and (9), it is not difficult to deduce a truth, that is, with increase of parameter $\alpha_{n-j-1, j}^{n-1}$ and $\alpha_{i, n-i-1}^{n-1}$, the QB-curve will infinitely approach edge $\overline{P_{n-j, j} P_{n-j-1, j+1}}$ and $\overline{P_{i, n-i} P_{i+1, n-i-1}}$, respectively, which indicates that the QB-curve could infinitely move to control polygon when the whole parameters $\alpha_{x,y}^{n-1} \rightarrow \infty$ simultaneously. Thus, what indicates that parameters $\alpha_{x,y}^{n-1}$ have good tension effect.

What we must mention is that parameters introduced by not adding degree of curve would possess a better computation efficiency and introduced by adding degree of curve have better approximation ability. Thus, the QB-curve constructed in this paper can retain two above advantages, that is QB-curve can infinitely approach control polygon without needing to add degree of curve. That is to say, the QB-curve can infinitely approach control polygon for any fixed degree $n(n = 3, 4, 5, 6, \dots)$.

Figure 3 will give some examples with regard to the locally adjusting effect of QB-curves of degree 4 with different parameters. In addition to the parameters shown in the figure, the other three parametric values are set as 0.

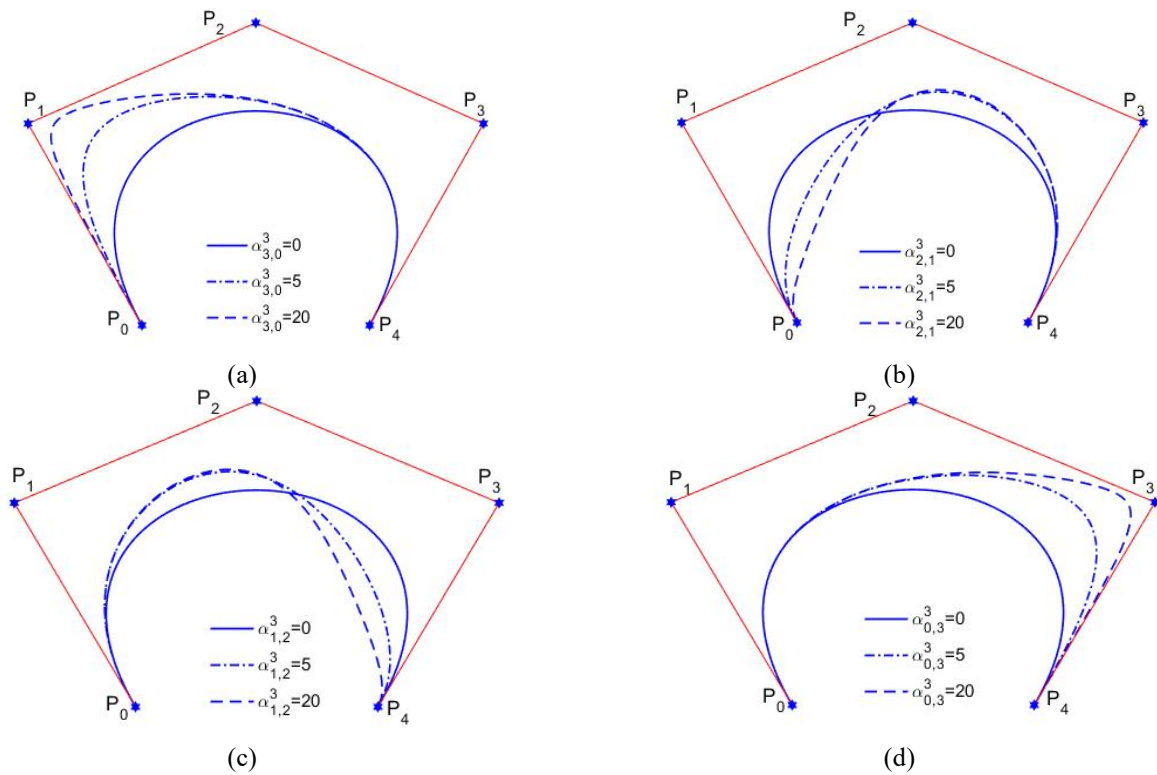


Figure 3 Adjusting Effect of Parameters on QB-Curves

5 ANALYSIS AND COMPARISON OF APPROXIMATION

In this subsection, we develop a method to roughly quantify the approaching degree between control polygon and target curves, which have important guiding significance.

As we known, if the basis has non-negativity and partition of unity, then the related curves have convex hull and affine invariance, which indicates that the curves absolutely lie inside its control polygon. And if the curves have beautiful property of approximation, then the curves can better mimic the behavior of control polygon. Therefore, it follows that the length of generated curve spanned by control polygon must smaller than control polygon. If the generated curves closer to control polygon, then the difference of the length between curve and control polygon will be smaller. Therefore, we develop a length method (L-Method) to quantify the degree of approximation.

Firstly, the length of target curves $R(t)$ spanned by control polygon P_0, P_1, \dots, P_n could be defined as follows:

$$L = \int_0^1 \sqrt{[R'_x(t)]^2 + [R'_y(t)]^2 + [R'_z(t)]^2} dt, \tag{10}$$

where $R_x(t), R_y(t)$, and $R_z(t)$ are parametric equations concerning the target curve $R(t)$ in three-dimension space.

Then, the expression for LM is defined by

$$LM = L / D \tag{11}$$

where

$$D = \sum_{j=1}^n d_j, d_i (i = 1, 2, 3, \dots, n) = |P_i - P_{i-1}|. \tag{12}$$

Remark 4 The value LM of the L-method can roughly measure the degree of approximation. The phenomenon which reflect from value LM is that the higher LM, the better approximation.

Figure 4 shows approaching effect of different curves.

Figure 4(a) and 4(c) show approaching effect of QB-curves with different parameters and Figure 4(b) and 4(d) show approaching comparison with method given in [10,18]. The blue, green and yellow line are quartic curve corresponding to QB-curves, literature and literature [10,18], respectively. The parametric values of references and are set under the case of the best approaching effect [10,18]. Therefore, from the images of Figure 4, it could conclude that the method proposed in this paper has better approaching effect than others. In order to better describe the conclusion shown in Figure 4, we apply the L-method to roughly quantify this results, see Table 1. Thus, we further sure that the QB-curves proposed in this paper have better approaching effect.

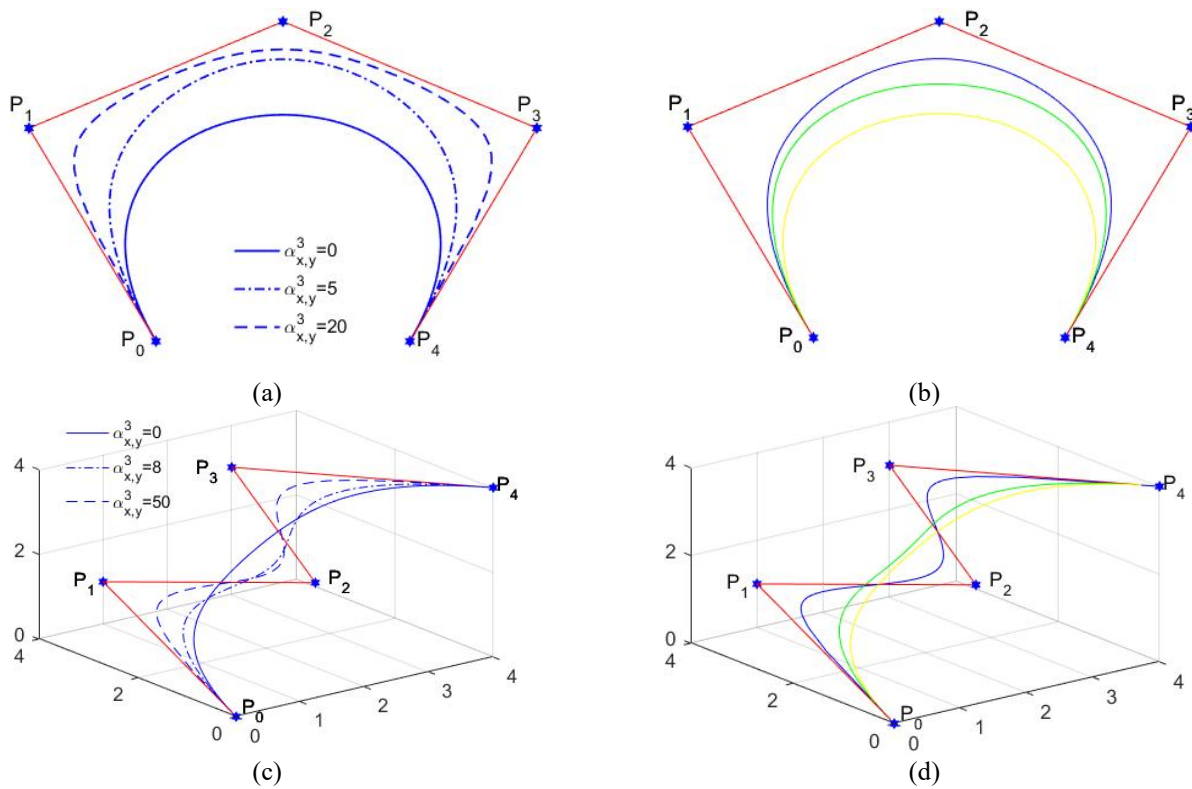


Figure 4 Approaching Comparison of Curves

Table 1 Comparison of LM Metrics with Different $\alpha_{x,y}^3$.

Figure	Method of this paper	LM
Figure (a)	$\alpha_{x,y}^3 = 0$	0.4683
Figure (a)	$\alpha_{x,y}^3 = 5$	0.5014
Figure (a)	$\alpha_{x,y}^3 = 20$	0.6349
Figure (c)	$\alpha_{x,y}^3 = 0$	0.5336
Figure (c)	$\alpha_{x,y}^3 = 8$	0.5623
Figure (c)	$\alpha_{x,y}^3 = 50$	0.6757
Figure (b)	$\alpha_{x,y}^3 = 20$	0.5608
Figure (b)	Literature[18]: $\omega = 0, \lambda_1 = \mu_1 = 5, \lambda_2 = \mu_2 = 10$	0.4683
Figure (b)	Literature[10]: $\lambda = 1$	0.4966
Figure (d)	$\alpha_{x,y}^3 = 50$	0.6757
Figure (d)	Literature[18]: $\omega = 0, \lambda_1 = \mu_1 = 5, \lambda_2 = \mu_2 = 10$	0.5336
Figure (d)	Literature[10]: $\lambda = 1$	0.5590

6 CONCLUSION

This study constructs quasi-Bessel curves based on multi-parameter n-order quasi-Bernstein bases, systematically investigating their construction methods, computational algorithms, shape control, approximation performance, and smooth stitching techniques. The following key conclusions are obtained:

- 1.The proposed Decastro-type angle-cutting algorithm enables efficient and stable computation of quasi-Bessel curves with varying degrees, providing a reliable method for their numerical implementation.
- 2.The parameters of quasi-Bessel curves exhibit excellent local adjustment and tension control effects. When fixing the control polygon, parameter adjustments enable localized modification of curve shapes. Moreover, increasing parameters allows the curve to infinitely approximate the control polygon without requiring order elevation, achieving optimal balance between computational efficiency and approximation capability.
- 3.The length-based quantization analysis method (L-Method) demonstrates that the quasi-Bessel curve outperforms existing similar methods in approximation quality, with quantitative LM value results further validating its superiority

in approximating control polygons.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

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