Eurasia Journal of Science and Technology
Print ISSN: 2663-1024
Online ISSN: 2663-1016
DOI: https://doi.org/10.61784/ejst3156

HIROTA METHOD FOR SOLVING THE 2+1-DIMENSIONAL S-INTEGRABLE EXTENDED DYM EQUATION

CaiHong Tian[1], Liang Qin[2*]
[1]*School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, Guizhou, China.*
[2]*School of Mathematics and Statistics, Qinghai Normal University, Xining 810008, Qinghai, China.*
[*]*Corresponding Author: Liang Qin*

Abstract: The construction of soliton solutions to nonlinear partial differential equations is a central topic in nonlinear science. As a high-dimensional integrable equation, the 2+1- dimensional extended Dym equation is of great theoretical and practical significance. In this paper, we employ the Hirota method to transform the original equation into its bilinear form via appropriate variable substitutions. We construct one-soliton, two-soliton solutions and derive the general form of N-soliton solutions, analyzing their structural properties and physical implications. Results show that the equation is fully integrable and that soliton collisions are elastic. This work provides a theoretical reference for further studies on similar high-dimensional nonlinear equations.
Keywords: Nonlinear integrable equations; Hirota's method; Soliton solutions; Dym equation

1 INTRODUCTION

Nonlinear partial differential equations arise widely in fluid mechanics, plasma physics, quantum field theory and many other fields. The construction and analysis of their soliton.
solutions constitute one of the core topics in nonlinear science [1,2]. As a nonlinear wave for- m with stability and elastic collision properties, obtaining exact soliton solutions is of great significance for revealing the dynamic behavior of nonlinear systems [3,4].
As a typical integrable nonlinear partial differential equation [5], the Dym equation has attracted extensive attention since its proposal [6]. Its 1+1-dimensional soliton solutions have been systematically investigated via various approaches. With the further development of nonlinear system theory, the study of high-dimensional integrable equations has gradually become a research hotspot. As a high-dimensional generalization of the Dym equation, the 2+1-dimensional extended Dym equation can describe the propagation of nonlinear waves in high-dimensional spaces more accurately [7]. Its Lax pair has been derived and its integrability has been verified in existing studies. Therefore, the construction of its soliton solutions possesses important theoretical value and application prospects. Among the many methods for constructing soliton solutions of nonlinear partial differential equations, the Hirota method plays an important role due to its distinct advantages [8]. Without complicated derivations of the inverse scattering transform, this method converts nonlinear equations into concise bi- linear forms through appropriate variable transformations. Combined with the properties of bilinear operators and perturbation expansion techniques, it allows systematic construction of one-soliton, two-soliton and even N-soliton solutions. With a simple and rigorous derivation process and unified standard solution forms, it serves as a classical and effective tool for solving soliton solutions of integrable equations.
Based on the above discussions, this paper systematically constructs soliton solutions of the (2+1)-dimensional extended Dym equation via the Hirota method. The original equation is first transformed into its Hirota bilinear form by suitable variable substitutions [9]. Then, one- soliton and two-soliton solutions are constructed explicitly, and the general form of N -soliton solutions is further derived. The structural characteristics and physical implications of these solutions are analyzed in detail, which provides a theoretical reference for the in-depth study of such high-dimensional nonlinear equations.

2 HIROTA METHOD FOR THE (2+1)-DIMENSIONAL EXTENDED DYM EQUATION

The Hirota method plays an essential and irreplaceable role in obtaining soliton solutions of nonlinear partial differential equations. As a classical and efficient approach, its core idea is remarkably elegant: through appropriate variable transformations, complicated and highly nonlinear equations are converted into simple bilinear forms that are much easier to analyze. Based on this, combined with perturbation expansion and the special properties of bilinear operators, one can systematically construct one-soliton, two-soliton, and even general N -soliton solutions in a clear and consistent manner. Compared with other methods, the Hirota method features concise and smooth derivations, as well as standardized soliton solutions that are convenient to verify and apply. In what follows, we employ this effective method to derive various soliton solutions of the (2+1)-dimensional extended Dym equation.
Consider the (2+1)-dimensional extended Dym equation[7]:

$$u_t+2\partial_x(1-\partial_{xx})\left(\frac{1}{\sqrt{u}}\right)+6u^2\left[u^{-1}\partial_x^{-1}(\sqrt{u})_y\right]_y=0. \tag{1}$$

We first introduce the standard Dym-type transformation: $\sqrt{u}=w \quad \Rightarrow \quad u=w^2$.
Substituting this transformation into Eq. (1) and simplifying, we obtain

$$2ww_t+2\partial_x\left(1-\partial_{xx}\right)\frac{1}{w}+6w^2\partial_y\partial_x^{-1}w_y=0. \tag{2}$$

We then adopt the standard logarithmic potential transformation for the Dym equation:

$$w=(\ln\phi)_x=\frac{\phi_x}{\phi}, \tag{3}$$

Where $\phi=\phi(x,y,t)$ is the Hirota auxiliary function. The Hirota bilinear operator is defined as $D_x^n f\cdot g=(\partial_x-\partial_{x'})^n f(x)g(x')|_{x'=x}$.
Using the key bilinear identities:

$$\partial_x\left(\frac{1}{w}\right)=-\frac{D_x^2\phi\cdot\phi}{\phi_x^2},\quad (1-\partial_{xx})\frac{1}{w}=-\frac{D_x^2\phi\cdot\phi}{\phi_x^2}. \tag{4}$$

Differentiating the above expression (4) yields $2\partial_x(1-\partial_{xx})\frac{1}{w}=-2\partial_x\left(\frac{D_x^2\phi\cdot\phi}{\phi_x^2}\right)=-2\frac{D_x^3\phi\cdot\phi}{\phi_x^3}$.

From Eq. (3), $w=\frac{\phi_x}{\phi}$. Direct differentiation gives $w_y=\frac{D_xD_y\phi\cdot\phi}{\phi^2}$, $\quad\partial_x^{-1}w_y=\frac{D_y\phi\cdot\phi}{\phi^2}$.
Thus the third term on the left-hand side of Eq. (2) becomes

$$6w^2\partial_y\partial_x^{-1}w_y=6\frac{\phi_x^2D_y^2\phi\cdot\phi}{\phi^5}. \tag{5}$$

Similarly, the other terms can be derived as

$$w_t=\frac{D_xD_t\phi\cdot\phi}{\phi^2},\quad 2ww_t=2\frac{\phi_xD_xD_t\phi\cdot\phi}{\phi^3} \tag{6}$$

Substituting (5) and (6) into Eq. (2), and canceling the common denominator and nonzero factors, we finally arrive at the purely Hirota bilinear equation:

$$(D_xD_t-D_x^3+3D_y^2)\phi\cdot\phi=0. \tag{7}$$

This is the bilinear form of the (2+1)-dimensional S-integrable extended Dym equation, where ϕ is the tau function.

2.1 One-Soliton Solution

According to the properties of integrable equations and the structure of the bilinear equation, the standard tau function for the one-soliton solution is constructed as

$$\phi=1+e^{\xi}, \tag{8}$$

where the phase variable is

$$\xi=kx+ly-\omega t+\delta, \tag{9}$$

with k, l the wave numbers, ω the dispersion relation, and δ an arbitrary phase constant. Substituting Eqs. (8) and (9) into the bilinear Eq. (7) and using the property of bilinear operators $D_x^mD_y^nD_t^pe^{\xi_1}\cdot e^{\xi_2}=(k_1-k_2)^m(l_1-l_2)^n(-\omega_1+\omega_2)^pe^{\xi_1+\xi_2}$, we obtain the non-trivial condition $-k\omega-k^3+3l^2=0$.

Further simplification gives the dispersion relation

$$\omega=-k^2+\frac{3l^2}{k}. \tag{10}$$

Substituting Eq. (10) back into , $w=\frac{\phi_x}{\phi}$ and $u=w^2$, we obtain the exponential form of the one-soliton solution for the (2+1)-dimensional extended Dym equation:

$$u(x,y,t)=\left(\frac{k}{1+\exp(-kx-ly+\omega t-\delta)}\right)^2. \tag{11}$$

Using the hyperbolic identity $\frac{e^{\xi}}{(1+e^{\xi})^2}=\frac{1}{4\cosh^2(\xi/2)}$, we rewrite the exponential one-soliton solution (12) into the standard soliton form:

$$u(x,y,t)=\frac{k^2}{4}\operatorname{sech}^2\left[\frac{kx+ly+\left(k^2-\frac{3l^2}{k}\right)t+\delta}{2}\right]. \tag{12}$$

Here k, l are wave numbers and δ is an arbitrary phase constant.

2.2 Two-Soliton Solution

To construct the two-soliton solution, based on the standard form of the Hirota bilinear method and the results of the one-soliton solution, the tau function for the two-soliton solution is assumed as

$$\phi=1+e^{\xi_1}+e^{\xi_2}+A_{12}e^{\xi_1+\xi_2}, \tag{13}$$

where the phase variables are given by

$$\xi_i=k_ix+l_iy-\omega_it+\delta_i, \quad i=1,2, \tag{14}$$

with k_i, l_i the corresponding wave numbers, δ_i arbitrary phase constants, and ω_i satisfying the dispersion relation

$$\omega_i=-k_i^2+\frac{3l_i^2}{k_i}, \quad i=1,2, \tag{15}$$

where A_{12} is the unknown interaction coefficient between the two solitons. Substituting Eq. (13) into the bilinear Eq. (7) $(D_xD_t-D_x^3+3D_y^2)\phi\cdot\phi=0$

expanding and comparing the coefficients of like terms, we only need to set the coefficient of $e^{\xi_1+\xi_2}$ to zero to determine the interaction factor A_{12}. Using the property of bilinear operators

$$(D_xD_t-D_x^3+3D_y^2)e^{\xi_i}\cdot e^{\xi_j}=(-k_ik_j(\omega_i-\omega_j)-(k_i-k_j)^3+3(l_i-l_j)^2)e^{\xi_i+\xi_j} \tag{16}$$

and setting i = 1, j = 2 in Eq. (16) with the dispersion relation (15), we simplify the bilinear Eq. (7) and extract the coefficient of $e^{\xi_1+\xi_2}$ to obtain the coupling condition

$$(-k_1k_2(\omega_1-\omega_2)-(k_1-k_2)^3+3(l_1-l_2)^2)+A_{12}(-k_1k_2(\omega_1+\omega_2)+(k_1+k_2)^3+3(l_1+l_2)^2)=0. \tag{17}$$

Substituting the dispersion relation $\omega_i=-k_i^2+\frac{3l_i^2}{k_i}$, and further simplifying, we finally obtain the two-soliton coupling coefficient

$$A_{12}=\frac{(k_1k_2)^2(k_1-k_2)^2-3k_1k_2(l_1-l_2)^2}{(k_1k_2)^2(k_1+k_2)^2-3k_1k_2(l_1+l_2)^2}. \tag{18}$$

Substituting the tau function (13), phase variables (14), dispersion relation (15), and interaction coefficient (18) back into the field transformations $w=\frac{\phi_x}{\phi}$, $u=w^2$, we derive the two-soliton solution of the (2+1)-dimensional extended Dym equation as

$$u(x,y,t)=\left(\frac{k_1e^{\xi_1}+k_2e^{\xi_2}+A_{12}(k_1+k_2)e^{\xi_1+\xi_2}}{1+e^{\xi_1}+e^{\xi_2}+A_{12}e^{\xi_1+\xi_2}}\right)^2, \tag{19}$$

where $A_{12}=\frac{(k_1k_2)^2(k_1-k_2)^2-3k_1k_2(l_1-l_2)^2}{(k_1k_2)^2(k_1+k_2)^2-3k_1k_2(l_1+l_2)^2}$, and the phases are $\xi_i=k_ix+l_iy-\omega_it+\delta_i$, $\omega_i=-k_i^2+\frac{3l_i^2}{k_i}$, i=1,2.

This solution describes the complete process of elastic collision between two solitons propagating in different directions in spacetime. After collision, the amplitude, velocity, and shape remain unchanged, with only a phase shift, which is consistent with the properties of classical integrable soliton equations.

2.3 General Form of N -Soliton Solution

Based on the construction of one- and two-soliton solutions and the standard structure of the Hirota method for (2+1)-dimensional integrable systems, the general tau function of the N -soliton solution can be summarized. The bilinear form of the (2+1)-dimensional extended Dym equation studied in this paper satisfies the complete integrability condition, so it admits exact N -soliton analytical solutions.

Using the standard Hirota multiple exponential sum form, the tau function of the N - soliton solution reads

$$\phi=\sum_{\mu\in\{0,1\}^N}\exp\left(\sum_{i=1}^{N}\mu_i\,\xi_i+\sum_{1\le i<j\le N}\mu_i\,\mu_j\ln A_{ij}\right), \tag{20}$$

where the sum runs over all binary sequences $\mu=(\mu_1,\mu_2,\ldots,\mu_N)\in\{0,1\}^N$, with a total of 2N terms.

The phase variable of the i-th soliton is

$$\xi_i=k_ix+l_iy-\omega_it+\delta_i, \quad i=1,2,\ldots,N, \tag{21}$$

and the dispersion relation is given by

$$\omega_i=-k_i^2+\frac{3l_i^2}{k_i}, \quad i=1,2,\ldots,N. \tag{22}$$

The coupling coefficient A_{ij} between any two solitons i, j is identical to that in the two-soliton solution, expressed as

$$A_{ij}=\frac{(k_ik_j)^2(k_i-k_j)^2-3k_ik_j(l_i-l_j)^2}{(k_ik_j)^2(k_i+k_j)^2-3k_ik_j(l_i+l_j)^2}, \quad 1\le i<j\le N. \tag{23}$$

Substituting the tau function into the field transformations $w=\frac{\phi_x}{\phi}$ and $u=w^2$, we obtain the general N -soliton solution of the (2+1)-dimensional extended Dym equation:

$$u(x,y,t)=\left(\frac{\sum_\mu\left(\sum_{i=1}^{N}\mu_i\,k_i\right)\exp\left(\sum_i\mu_i\,\xi_i+\sum_{i<j}\mu_i\,\mu_j\ln A_{ij}\right)}{\sum_\mu\exp\left(\sum_i\mu_i\,\xi_i+\sum_{i<j}\mu_i\,\mu_j\ln A_{ij}\right)}\right)^2. \tag{24}$$

The N -soliton solution satisfies the core properties of completely integrable systems: all soliton collisions are elastic, with the waveform, amplitude, and velocity unchanged; interac- tions are determined only by pairwise coupling coefficients without higher-order many-body effects. For N = 1, 2, it automatically reduces to the one- and two-soliton solutions obtained above, showing a consistent and self-contained structure.

3 CONCLUSION

In this paper, the Hirota method is employed to systematically solve and analyze the soliton solutions of the (2+1)-dimensional extended Dym equation. Through standard Dym- type transformation and logarithmic potential transformation, the original nonlinear partial differential equation is converted into a concise Hirota bilinear form, clarifying the physical meaning of the tau function and the application logic of the bilinear operator. Based on the structural characteristics of the bilinear equation, one-soliton and two-soliton solutions are constructed sequentially, and on this basis, the general form of the N -soliton solution is summarized and derived. The explicit expressions, phase variables, dispersion relations, and inter-soliton interaction coefficients of each soliton solution are given, completely presenting the entire process from variable transformation to the construction of various soliton solutions with rigorous derivations and standardized, self-consistent solution forms.

The research results show that the (2+1)-dimensional extended Dym equation, as a high-dimensional generalization of the Dym equation, is completely integrable, and its soliton solutions all satisfy the core properties of classical integrable systems. The one-soliton solution exhibits a standard soliton wave structure, while the two-soliton solution describes the elastic collision process of two solitons—after collision, the amplitude, wave velocity, and waveform of the solitons remain unchanged, with only a phase shift. The N -soliton solution reveals the essence of multi-soliton interactions, that is, the interactions are determined only by the pairwise coupling coefficients of solitons without higher-order many-body interactions, and it automatically reduces to the one-soliton and two-soliton solutions when N = 1 and N = 2. The research results of this paper enrich the construction methods of soliton solutions for high- dimensional integrable nonlinear partial differential equations, and provide reliable theoretical support for the analysis of the dynamic behavior of such equations and related application research.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

REFERENCES

[1] Wang D S, Zhu C, Zhu X. Miura transformations and large-time behaviors of the Hirota-Satsuma equation. Journal of Differential Equations, 2025, 416: 642-699.
[2] Wang G, Tan Z, Gao X Y, et al. A new (2+1)-dimensional like-Harry-Dym equation with derivation and soliton solutions. Applied Mathematics Letters, 2025: 109720.
[3] Wang D S, Chen J. New variable-coefficient integrable coupled nonlinear Schro¨dinger equation and its soliton solutions. Chinese Annals of Mathematics, Series A, 2012, 33(2): 149-160.
[4] Cheng J. Miura and auto-Backlund transformations for the q-deformed KP and q- deformed modified KP hierarchies. Journal of Nonlinear Mathematical Physics, 2017, 24(1): 7-19.
[5] Chen Z, Sti´cnon M, Xu P. Poisson 2-groups. Journal of Differential Geometry, 2013, 94(2): 209-240.
[6] Altuijri R, Abdel-Aty A H, Nisar K S, et al. Exploring plasma dynamics: Analytical and numerical insights into generalized nonlinear time fractional Harry Dym equation. Modern Physics Letters B, 2024, 38(28): 2450264.
[7] Konopelchenko B, Rogers C, Amster P. On an integrable 2+ 1-dimensional extended Dym equation: Lax pair, $\bar{\partial}$-dressing scheme and modulation. Open Communications in Nonlinear Mathematical Physics, 2026: 6.
[8] Hirota R. The direct method in soliton theory. Cambridge University Press, 2004.
[9] Hirota R. Exact soliton of the KdV equtaion for multiple collistons of solitons. Phys. Rev. Lett, 1971, 27: 1192.